Entanglement versus mixedness for coupled qubits under a phase damping channel

E. S. Cardoso*, M. C. de Oliveira†, K. Furuya‡
Instituto de Física Gleb Wataghin, Universidade Estadual de Campinas, 13083-970, Campinas - SP, Brazil.
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Quantification of entanglement against mixing is given for a system of coupled qubits under a phase damping channel. A family of pure initial joint states is defined, ranging from pure separable states to maximally entangled state. An ordering of entanglement measures is given for well defined initial state amount of entanglement.
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I. INTRODUCTION

Quantum entanglement is an essential resource for many quantum communication protocols, such as teleportation [1,2] and dense coding [3,4], allowing an information processing efficiency, which is otherwise unattainable through classical protocols. As such the efficiency of those quantum protocols relies on the ability to isolate the encoding system, being strongly decreased when the system is coupled to an external environment, attaining thus the classical limits, when the quantum system is found in a separable state [5]. The relation of entanglement against separability for mixed entangled states has generated a considerable literature, with many proposals for quantifying it (see for example [6–10] and references therein). Bipartite quantum systems with Hilbert space of dimension $2 \otimes 2$ (coupled qubits) have been exhaustively investigated in order to achieve a precise quantification of entanglement against mixing. Particularly a valuable necessary and sufficient condition for separability of coupled qubits has been given by Peres [7] and Horodecki [8] in terms of the *positivity* of the partial transpose of the system density matrix, which sets a boundary for comparison to many proposed entanglement measures, such as entanglement of formation, relative entropy of distillation, and relative entropy of entanglement. A very useful paper has appeared recently [9] relating the ordering of many entanglement measures in relation to the degree of mixedness. This paper reinforce and extends the discussion presented in Ref. [6] on maximally entangled mixed states (MEMS), which are states that for a given mixedness achieve the greatest possible entanglement. More recently the imbalance between the sensitivity of common state measures, such as fidelity trace distances, concurrence, tangle and von Neumann entropies when acted on by a depolarizing channel have also been investigated [10]. It was noticed that the size of the imbalance depends intrinsically of the state tangle and of the state purity. The results of Refs. [9,10] were derived for arbitrary entangled mixed states - randomly generated bipartite matrices respecting the structure of positive semidefinite operators. An actual bipartite interacting system has its entanglement constrained by its dynamics and thus (for certain given initial states) they never reach the discussed MEMS. The system's dynamics constrains the degree of entanglement and mixedness to a bounded range. Only the MEMS dynamically connected to the system state are important for setting reference states, and thus only those states are valid for entanglement quantification in terms of distance measures. Those MEMS can be computed by a certain combination of the reduced density matrix of the coupled qubits. It is thus of central importance to analyze the amount of entanglement against mixing present in a quantum system due to a process of deterministic entanglement formation [11] in a noisy channel.

In this paper we analyze the degree of entanglement against mixing for a dynamical system composed of two coupled qubits under the phase damping channel. While amplitude damping is certainly the most important source of noise for light field states qubit encoding, the phase damping model describes more appropriately noise over an encoding system composed of internal atomic (ionic) states or even for internal quantum dots states [12–16]. The phase damping channel is particularly interesting in analyzing the degree of entanglement against mixing because it truly induces decoherence without amplitude relaxation effects [5]. We compare many entanglement measures as a function of the joint state purity and discuss how do they relate to each other for the specific dynamical system considered. More specifically we compare concurrence and negativity with Bures distance entanglement measures. While concurrence and negativity are able to quantify the amount of entanglement present in a mixed state, they are not able to distinguish states. The Bures distance entanglement measure, on the other hand, is able to distinguish states and thus can be used to define a ordering of entanglement measures. In its definition however, a deep analysis

*esc@ifi.unicamp.br
†marcos@ifi.unicamp.br
‡furuya@ifi.unicamp.br

must be made on the reference state, from where the distance measure is taken. We have considered both mixed and pure reference states, since both situations allows inspection of many distinct entanglement features. In Sec. II we present the model of coupled qubits under the phase damping channel. In Sec. III we present the measures used to infer the amount of entanglement and mixing through the paper. In Sec. IV we define entanglement measures in terms of Bures distance to maximally mixed and maximally entangled states. In Sec. V we compare all the discussed entanglement measures against mixedness and comment a possible measure ordering for the proposed model. Finally, Sec. VI concludes the paper.

II. COUPLED QUBITS UNDER PHASE DAMPING CHANNEL

The system considered is constituted of two qubits coupled by a phase interaction under the effect of a common environment constituted by N harmonic oscillators. Since we are interested in the effect of the environment over the purity of the coupled qubit system, we consider a number conserving interaction to the harmonic oscillator set. A straightforward application of the procedures developed here is envisaged for qubits encoded in internal states of trapped ions in the proposal of Cirac and Zoller [12–15] for quantum computation. Following the DiVincenzo criteria [17], quantum gate operations must be shorter than the decoherence time. Past results have shown that decoherence in trapped ions appears without energy exchange [16] and thus cannot be explained by the physical processes that take in consideration the energy exchange as a source of decoherence, and thus the inadequacy of the amplitude damping model. Our model Hamiltonian writes

$$H = H_S + H_R + H_I, \tag{1}$$

with

$$H_S = \omega_1 S_{1z} + \omega_2 S_{2z} + \mu_{12} S_{1z} S_{2z}, \tag{2}$$

$$H_R = \hbar\omega_1 \sum_{k=1}^{N} \widetilde{\omega}_k \left(n_k + \frac{1}{2} \right) + 2\hbar\mu \sum_{i<j}^{N} n_i n_j. \tag{3}$$

$$H_I = \sum_{k=1}^{N} n_k \left(\mu_1 S_{1z} + \mu_2 S_{2z} \right), \tag{4}$$

with

$$\widetilde{\omega}_k = \frac{\omega_k}{\omega_1}$$

In this model the reservoir, together with the proposed interaction is responsible by decoherence without energy damping.

We shall investigate how the proposed model describes the evolution from a pure maximally entangled state to a separable state. For that we base our discussion on some entanglement measures previously discussed [6,9,18]. Firstly consider that the two qubit states are prepared in an entangled pure state in contact to a reservoir prepared as such

$$\begin{aligned} |\psi(0)\rangle = {} & \frac{1}{\sqrt{2\varepsilon(\varepsilon-1)+1}} (\varepsilon|+,+\rangle + (1-\varepsilon)|-,-\rangle) \\ & \otimes \prod_{i=1}^{N} |\alpha_i\rangle. \end{aligned} \tag{5}$$

The degree of entanglement of the initial state is a function of ε, which varies from 0 to 1. The maximally entangled pure initial state is reached for $\varepsilon = 0.5$, while pure separable states are obtained for $\varepsilon = 0$ and 1. The evolved joint state given by the evolution of the state (5) due to the Hamiltonian (1)-(4) reads

$$\rho(t) = \frac{1}{(2\varepsilon(\varepsilon-1)+1)} \sum_{n_1 \dots n_N} \sum_{n_1' \dots n_N'} \prod_{j=1}^{N} \frac{\alpha_j^{n_j} \alpha^{* n_j'}}{\sqrt{n_j! n_j'!}} \{ \varepsilon^2 e^{-i\{\phi_{1j}(t) - \phi_{1j}'(t)\}} \mid +,+,n_1,\dots,n_N\rangle\langle +,+,n_1',\dots,n_N' \mid +$$
$$+(1-\varepsilon)^2 e^{-i\{\phi_{2j}(t) - \phi_{2j}'(t)\}} \mid -,-,n_1,\dots,n_N\rangle\langle -,-,n_1',\dots,n_N' \mid +$$
$$+\varepsilon(1-\varepsilon) e^{-i(\omega_1+\omega_2)t} e^{-i\{\phi_{1j}(t) - \phi_{2j}'(t)\}} \mid +,+,n_1,\dots,n_N\rangle\langle -,-,n_1',\dots,n_N' \mid +$$
$$+\varepsilon(1-\varepsilon) e^{+i(\omega_1+\omega_2)t} e^{+i\{\phi_{1j}'(t) - \phi_{2j}(t)\}} \mid -,-,n_1,\dots,n_N\rangle\langle +,+,n_1',\dots,n_N' \mid\} \qquad (6)$$

with

$$\phi_{1j}(t) = +n_j(\frac{\mu_1+\mu_2}{2})t + \omega_1 \widetilde{\omega}_j t(n_j + \frac{1}{2})$$
$$+2\mu\omega_1 t \sum_{k>j} \widetilde{\omega}_k n_k \widetilde{\omega}_j n_j \qquad (7)$$

$$\phi_{2j}(t) = -n_j(\frac{\mu_1+\mu_2}{2})t + \omega_1 \widetilde{\omega}_j t(n_j + \frac{1}{2})$$
$$+2\mu\omega_1 t \sum_{k>j} \widetilde{\omega}_k n_k \widetilde{\omega}_j n_j \qquad (8)$$

and

$$\phi_{1j}'(t) = \phi_{1j}(n \longrightarrow n')$$
$$\phi_{2j}'(t) = \phi_{2j}(n \longrightarrow n'). \qquad (9)$$

When we consider a particular case of a resonant bath, where all the ω_j's of the bath are the same, one can obtain a closed expression for the reduced density matrix of the coupled qubits, which then writes as

$$\rho_\varepsilon = \begin{pmatrix} \frac{\varepsilon^2}{(2\varepsilon(\varepsilon-1)+1)} & 0 & 0 & \frac{\varepsilon(1-\varepsilon)}{(2\varepsilon(\varepsilon-1)+1)} A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\varepsilon(1-\varepsilon)}{(2\varepsilon(\varepsilon-1)+1)} A^* & 0 & 0 & \frac{(1-\varepsilon)^2}{(2\varepsilon(\varepsilon-1)+1)} \end{pmatrix}, \qquad (10)$$

which clearly has the structure of a mixed nonmaximally mixed state with

$$A = e^{-i(\omega_1+\omega_2)t} e^{\sum_{j=1}^{N} |\alpha_j|^2 [e^{-i(\mu_1+\mu_2)t} - 1]}$$
$$= e^{-i(\omega_1+\omega_2)t} e^{\widetilde{N}[e^{-i(\mu_1+\mu_2)t} - 1]}, \qquad (11)$$

and $\widetilde{N} \equiv \sum_{j=1}^{N} |\alpha_j|^2$. We then identify the typical operation sum structure of the phase damping channel

$$\rho_\varepsilon = E_0 \rho_\varepsilon^0 E_0^\dagger + E_1 \rho_\varepsilon^0 E_1^\dagger, \qquad (12)$$

with

$$E_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-\gamma} \end{pmatrix} \quad E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma} \end{pmatrix}, \qquad (13)$$

and

$$\rho_\varepsilon^0 = \begin{pmatrix} \frac{\varepsilon^2}{(2\varepsilon(\varepsilon-1)+1)} & 0 & 0 & \frac{\varepsilon(1-\varepsilon)e^{-i\phi(t)}}{(2\varepsilon(\varepsilon-1)+1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\varepsilon(1-\varepsilon)e^{i\phi(t)}}{(2\varepsilon(\varepsilon-1)+1)} & 0 & 0 & \frac{(1-\varepsilon)^2}{(2\varepsilon(\varepsilon-1)+1)} \end{pmatrix}, \qquad (14)$$

where $\gamma \equiv 1 - e^{2\widetilde{N}[\cos(\mu_1+\mu_2)t - 1]} = 1 - |A|^2$, and $e^{-i\phi(t)} \equiv e^{-i[(\omega_1+\omega_2)t + \widetilde{N}\sin(\mu_1+\mu_2)t]} = A/|A|$.

III. ENTANGLEMENT MEASURES AND MIXING

We shall refer to the state degree of mixing through the linear entropy

$$\delta_{12} = \frac{d}{d-1}\left[1 - \mathrm{Tr}(\rho_\varepsilon^2)\right], \tag{15}$$

where d is the dimension of the system Hilbert space. The linear entropy ranges from 0 (for pure states) to 1 (for maximally mixed states $\rho_{MM} = I_d/d$). It has been determined that arbitrary bipartite states whose linear entropy $\delta_{12} \geq d(d-2)/(d-1)^2$ are separable [9]. For the coupled qubits system that means that states whose $\delta_{12} \geq 8/9$ are certainly separable [9]. For the state considered here the linear entropy explicitly reads

$$\begin{aligned}\delta_{12}(\varepsilon,t) &= \frac{4}{3}\left\{\frac{2\varepsilon^2(1-\varepsilon)^2(1-|A|^2)}{[\varepsilon^2+(1-\varepsilon)^2]^2}\right\} \\ &= \frac{4}{3}\left\{\frac{2\varepsilon^2(1-\varepsilon)^2\gamma}{[\varepsilon^2+(1-\varepsilon)^2]^2}\right\}.\end{aligned} \tag{16}$$

Since $0 \leq \varepsilon \leq 1$ and $1-\varepsilon^2(1-\varepsilon)^2 \leq \gamma \leq 1$, it is immediate to see that the system state never reaches the maximally mixed state and that $\max\delta_{12}(t) = 2/3$, which is well below the limit of 8/9 given for bipartite qubit states. Although the state is separable, as we will shortly discuss.

An important measure of entanglement is the negativity of the state calculated as $(C^2 \otimes C^2)$ [9,18]. This last criterion is related to the separability of the state considering that the state is separable if the partially transposed state is also a valid quantum state, that is a positive semidefinite operator [9,18]. The partial transposition of a non-separable state presents one negative eigenvalue and thus we need to follow the eigenvalues of the partially transposed joint state. For the calculation of the negativity we have considered the definition [9,18]

$$N(\rho,t) = 2\max\{0, -\lambda_{neg}(t)\}. \tag{17}$$

For the initial state (5) here considered

$$\begin{aligned}N(\rho_\varepsilon,t) &= \frac{2\varepsilon(1-\varepsilon)|A|}{\varepsilon^2+(1-\varepsilon)^2} \\ &= \frac{2\varepsilon(1-\varepsilon)\sqrt{1-\gamma}}{\varepsilon^2+(1-\varepsilon)^2}.\end{aligned} \tag{18}$$

Notice that for $t = 0$

$$N(\rho_\varepsilon,0) = \frac{2\varepsilon(1-\varepsilon)}{\varepsilon^2+(1-\varepsilon)^2}, \tag{19}$$

which is maximal for $\varepsilon = 1/2$. $N(\rho_\varepsilon,0)$ is exactly the coherence of the initial pure state given by (5) and its maximal value represents the maximally entangled pure state given by (5) for $\varepsilon = 1/2$. For this special case

$$N(\rho_{1/2},t) = e^{\widetilde{N}[\cos((\mu_1+\mu_2)t)-1]} = \sqrt{1-\gamma}. \tag{20}$$

Although $N(\rho_{1/2},t)$ does not change sign it gets rapidly closer to zero for $\widetilde{N} \gg 1$. Only for $N(\rho_{1/2},t) = 0$ (or $\gamma = 1$) the system is *separable*.

Another important measure of entanglement which has an exact analytic expression for coupled qubits is the entanglement of formation [6,18,19]. It is defined as

$$E_F = h\left(\frac{1}{2}\left[1+\sqrt{1-C(\rho)^2}\right]\right). \tag{21}$$

being h and the concurrence $C(\rho)$ defined as [6,18,19]

$$h_x = -x\log_2 x - (1-x)\log_2(1-x), \tag{22}$$

$$C(\rho) \equiv \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}, \tag{23}$$

where λ_i are the eigenvalues of $\rho\sigma_2 \otimes \sigma_2\rho^*\sigma_2 \otimes \sigma_2$, where σ_2 is the Pauli σ_y-spin matrix. For the above state (5),

$$C(\rho_\varepsilon) = \sqrt{1 - \frac{2\varepsilon^2(1-\varepsilon)^2(1-|A|^2)}{[\varepsilon^2 + (1-\varepsilon)^2]^2}}. \tag{24}$$

For the maximally entangled state

$$\begin{aligned} C(\rho_{1/2}) &= \sqrt{\frac{1}{2}\left\{1 + e^{2\tilde{N}[cos((\mu_1+\mu_2)t)-1]}\right\}} \\ &= \sqrt{1 - \frac{\gamma}{2}} \end{aligned} \tag{25}$$

For the special dynamical system considered the above mentioned measures are all monotonic functions of each other. For example we can write all the other measures as a function of the negativity $N(t)$ as

$$\delta_{12}(\varepsilon, t) = \frac{2}{3}\left(N^2(\rho_\varepsilon, 0) - N^2(\rho_\varepsilon, t)\right), \tag{26}$$

$$C(\rho_\varepsilon) = \sqrt{1 - \frac{1}{2}\left(N^2(\rho_\varepsilon, 0) - N^2(\rho_\varepsilon, t)\right)}, \tag{27}$$

which for $\varepsilon = 1/2$ writes

$$\delta_{12}(1/2, t) = \frac{2}{3}\left(1 - N(\rho_{1/2}, t)^2\right) \tag{28}$$

$$C(\rho_{1/2}) = \sqrt{\frac{1}{2}\left(1 + N(\rho_{1/2}, t)^2\right)}. \tag{29}$$

IV. DISTANCE AS ENTANGLEMENT MEASURES

While the above considered entanglement measures are capable to quantify the amount of entanglement present in the quantum state (10) they lack an interpretative meaning. It is possible to define an entanglement measure $\mathcal{E}(\rho, \sigma)$ of a quantum state ρ as a distance measure between the quantum state ρ and a reference state σ. The distance must be minimized over all the dynamically connected reference states σ. For example, the Bures distance [20] was recently identified as a possible quantification of entanglement [21–23], for one and two parties states, respectively. The Bures distance is defined as

$$d_B(\rho, \sigma) = (2 - 2\sqrt{\mathcal{F}(\rho, \sigma)})^{1/2}, \tag{30}$$

where $\mathcal{F}(\rho, \sigma)$ is the Uhlmann Fidelity [20] between any two quantum states ρ and σ: $\mathcal{F}(\rho, \sigma) = \{Tr[(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}]\}^2$, ranging from 0 to 1. In such a case $d_B(\rho, \sigma)$ must be minimized over the set the possible referential σ states.

Two choices of the reference state can be made, from which the distance measure definition as an entanglement measure will be dependent: (i) a mixed state reference, and (ii) a pure state reference. A mixed state reference is a natural choice, since it was proven that a pure reference state does not allows that the distance-based entanglement measure be an entanglement monotone (see [21,22]), once it can always be increased by appropriate local operations on ρ. On the other hand, the redundancy of possible mixed reference states, as we will discuss in what follows, and the appealing physical meaning that a pure reference state offers, make it interesting for comparison with the previously described entanglement measures. In what follows we will consider both (i) and (ii) situations, and show how do they relate to each other.

A. Mixed reference state

For a mixed reference state, the closer ρ_ε is from the reference σ, the less pure it will be and thus the state will be less entangled. That means that for a mixed reference state σ_m, the Bures distance itself can be regarded as an entanglement measure [22]

$$\mathcal{E}(\rho_\varepsilon,\sigma_m)=\min_{\sigma\in\mathcal{D}}\frac{1}{2}d_B(\rho_\varepsilon,\sigma_m)^2, \tag{31}$$

where $\mathcal{D}$ is the set of all separable bipartite states of the system. $\mathcal{E}(\rho_\varepsilon,\sigma_m)$ was numerically calculated and it will be presented in next section. For the maximally entangled initial state considered here ($\varepsilon=1/2$), $\mathcal{E}(\rho_{1/2},\sigma_m)$ has a simple expression. In this situation the reference state which gives the minimal distance is the following state:

$$\sigma_m=\left(\begin{array}{cccc} \frac{1}{2} & 0 & 0 & \frac{e^{-i(\omega_1+\omega_2)t}}{2e^{2\widetilde{N}}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{e^{i(\omega_1+\omega_2)t}}{2e^{2\widetilde{N}}} & 0 & 0 & \frac{1}{2} \end{array}\right). \tag{32}$$

For this state the fidelity is

$$\mathcal{F}(\rho_{1/2},\sigma_m)=\frac{1}{4}\left[\sqrt{1+\beta-\alpha}+\sqrt{1+\beta+\alpha}\right]^2, \tag{33}$$

and the entanglement measure will be

$$\mathcal{E}(\rho_{1/2},\sigma_m)=1-\frac{1}{2}\left[\sqrt{1+\beta-\alpha}+\sqrt{1+\beta+\alpha}\right], \tag{34}$$

with

$$\alpha=\sqrt{(-1-\beta)^2-(-1+|A|^2)(-1+4\,|X|^2)} \tag{35}$$

$$\beta=A^*X+AX^*, \tag{36}$$

and

$$X=e^{-i(\omega_1+\omega_2)t}e^{-2\widetilde{N}} \tag{37}$$

As such the entanglement measure ranges from 0 for a separable state to $1-\frac{1}{\sqrt{2}}\sqrt{1+e^{-2\widetilde{N}}}$ for a maximally entangled state.

Notice that a natural choice would be to use as a reference separable state the maximally mixed state

$$\sigma_{mm}=\frac{I}{4}. \tag{38}$$

However, such a state does not belong to the set of separable states attained by the quantum system here considered in the chosen conditions. We will consider this state for comparison with the result obtained for an actual state σ belonging to $\mathcal{D}$ and show that, despite their distinction, the distance measures introduced above approximate each other as $\widetilde{N}\to\infty$. With respect to the *maximally mixed state* (38) the fidelity of the quantum state gives

$$\mathcal{F}(\rho_{1/2},\sigma_{mm})=\frac{1}{4}\left[\sqrt{1-|A|}+\sqrt{1+|A|}\right]^2, \tag{39}$$

for A given by (11). The entanglement measure gives

$$\mathcal{E}(\rho_{1/2},\sigma_{mm})=1-\frac{1}{2}\left[\sqrt{1-|A|}+\sqrt{1+|A|}\right], \tag{40}$$

thus ranging from zero to $1-\frac{1}{2}\left(\sqrt{1-e^{-2\widetilde{N}}}+\sqrt{1+e^{-2\widetilde{N}}}\right)$ for a separable state, and from zero to $1-\frac{1}{\sqrt{2}}$ for a maximally entangled state. Notice that as $\widetilde{N}\to\infty$, then $\mathcal{E}(\rho,\sigma_{mm})\to\mathcal{E}(\rho,\sigma_m)$, despite being for different reference states.

B. Pure reference state

For a pure reference state the Bures distance measure must be minimized over all the dynamically connected reference pure states, thus satisfying the following criteria

$$\begin{aligned} Tr_1\left\{\rho_{12}^P(t)\right\} &\equiv \rho_2(t) \\ Tr_2\left\{\rho_{12}^P(t)\right\} &\equiv \rho_1(t), \end{aligned} \tag{41}$$

and the Uhlmann fidelity reduces to the usual fidelity, $\mathcal{F}(\rho(t),\sigma_p) = Tr\{\rho(t)\sigma_p\}$. Now the Bures distance writes simply as

$$d_B(\rho,\sigma) = (2 - 2\mathcal{F}(\rho,\sigma))^{1/2}. \tag{42}$$

Remark that the relative entropy related to the pure state σ_p is defined as

$$\begin{aligned} E_r(\rho,\sigma_p) &= Tr\left\{\sigma_p \log_2 \sigma_p - \sigma_p \log_2 \rho(t)\right\} \\ &= -Tr\left\{\sigma_p \log_2 \rho(t)\right\}, \end{aligned} \tag{43}$$

in binary units or

$$E_r(\rho,\sigma_p) = -Tr\left\{\sigma_p \ln \rho(t)\right\}, \tag{44}$$

in natural units. Since $\rho(t) = 1 - (1-\rho(t))$, such that $Tr\{1-\rho(t)\} < 1$, thus $\ln[1-(1-\rho(t))] = -[(1-\rho(t)) + (1-\rho(t))^2/2 + ...]$ and the relative entropy can be written to first order in $(1-\rho(t))$ as

$$\begin{aligned} E_r(\rho,\sigma_p) &\approx 1 - Tr\left\{\sigma_p \rho(t)\right\} \\ &= 1 - \mathcal{F}(\rho,\sigma_p). \end{aligned} \tag{45}$$

That means that the Uhlmann fidelity of the quantum state $\rho(t)$ to a pure reference state σ_p corresponds to one minus the linearized relative entropy. Now defining the entanglement measure as

$$\mathcal{E}(\rho,\sigma_p) = 1 - \min_{\sigma_p \in \mathcal{D}} \frac{1}{2} d_B(\rho,\sigma_p)^2, \tag{46}$$

we obtain

$$\mathcal{E}(\rho,\sigma_p) = \mathcal{F}(\rho,\sigma_p) = 1 - E_r(\rho,\sigma_p), \tag{47}$$

which shows in a nice way how the distance entanglement measure relates to the relative entropy for pure reference states.

The reference state is simply a purified version of the studied quantum state. It could in fact represent a whole family of states, if we have not considered that the system dynamics restricts the possible reference states as follows. Since the system dynamics does not allow energy transference between subsystems the initial unpopulated subspace $(|+,-\rangle, |-,+\rangle)$ does not participate in the choice of the pure reference state. Moreover, the intrinsic system dynamics must be included in the reference state. The perfect choice is $\sigma_p = \rho_\varepsilon^0$ given by Eq. (14).

Thus in this case the fidelity of the system state to the pure reference state (σ_p) and thus the entanglement measure writes as

$$\begin{aligned} \mathcal{E}(\rho_\varepsilon,\sigma_p) &= \frac{\left[\varepsilon^4 + (1-\varepsilon)^4 + 2\varepsilon^2(1-\varepsilon)^2|A|\right]}{\left[\varepsilon^4 + (1-\varepsilon)^4 + 2\varepsilon^2(1-\varepsilon)^2\right]} \\ &= 1 - \frac{2\varepsilon^2(1-\varepsilon)^2(1-|A|)}{[\varepsilon^2 + (1-\varepsilon)^2]^2}, \end{aligned} \tag{48}$$

or

$$\mathcal{E}(\rho_\varepsilon,\sigma_p) = 1 - \frac{1}{2} N(\rho_\varepsilon,0)\left(N(\rho_\varepsilon,0) - N(\rho_\varepsilon,t)\right). \tag{49}$$

For $\varepsilon = 1/2$

$$\mathcal{E}(\rho_{1/2},\sigma_p) = \frac{1}{2}\left\{1+e^{\widetilde{N}[\cos((\mu_1+\mu_2)t)-1]}\right\}$$
$$= \frac{1}{2}\left\{1+N(\rho_{1/2},t)\right\}, \quad (50)$$

thus ranging from $\frac{1}{\sqrt{2}}\sqrt{1+e^{-2\widetilde{N}}}$ to 1 for a separable state and a maximally entangled state, respectively. Notice that this range is displaced in $\frac{1}{\sqrt{2}}\sqrt{1+e^{-2\widetilde{N}}}$ in relation to the range for $\mathcal{E}(\rho,\sigma_m)$.

As such the relative entropy writes as

$$E_r(\rho_\varepsilon,\sigma_p) = \frac{1}{2}N(\rho_\varepsilon,0)\left(N(\rho_\varepsilon,0)-N(\rho_\varepsilon,t)\right) \quad (51)$$

and

$$E_r(\rho_{1/2},\sigma_p) = \frac{1}{2}\left(1-N(t)\right). \quad (52)$$

We remark that for pure states $E_F = E_r$, and for a general mixed state $E_F \geq E_r$.

V. ENTANGLEMENT MEASURES ORDERING

Before we compare the entanglement measures by varying the degree of mixing we observe that the state purity and the concurrence can be related to the linearized relative entropy and the distance entanglement measure as

$$\delta_{12}(\rho_\varepsilon,t)N^2(\rho_\varepsilon,0) = \frac{2d}{d-1}\left\{E_r(\rho_\varepsilon,\sigma_p)\frac{2d}{d-1}\left(\mathcal{E}(\rho_\varepsilon,\sigma_p)-1\right)+\frac{1}{2}N(\rho_\varepsilon,0)\left[E_r(\rho_\varepsilon,\sigma_p)\left(1+N(\rho_\varepsilon,0)\right)\right.\right.$$
$$\left.\left.+\left(\mathcal{E}(\rho_\varepsilon,\sigma_p)-1\right)\left(1-N(\rho_\varepsilon,0)\right)\right]\right\} \quad (53)$$
$$C(\rho_\varepsilon)^2N^2(\rho_\varepsilon,0) = \left(\mathcal{E}(\rho_\varepsilon,\sigma_p)-1\right)^2+E_r^2(\rho_\varepsilon,\sigma_p)+\mathcal{E}(\rho_\varepsilon,\sigma_p)N(\rho_\varepsilon,0)\left(1+N(\rho_\varepsilon,0)\right)$$
$$+E_r(\rho_\varepsilon,\sigma_p)N(\rho_\varepsilon,0)\left(1-N(\rho_\varepsilon,0)\right), \quad (54)$$

Notice that for the maximally entangled state at $\varepsilon = 1/2$ the state purity can be related to the linearized relative entropy and the distance entanglement measure as

$$\delta_{12}(\rho_{1/2},t) = \frac{2d}{d-1}\mathcal{E}(\rho_{1/2},\sigma_p)E_r(\rho_{1/2},\sigma_p), \quad (55)$$

while the concurrence can be written as

$$C(\rho_{1/2})^2 = \mathcal{E}^2(\rho_{1/2},\sigma_p)+E_r^2(\rho_{1/2},\sigma_p), \quad (56)$$

thus defining a quite interesting triangular relation between the concurrence the distance entanglement measure and the relative entropy.

The relation between entanglement and mixing measures can be used to define an entanglement measure ordering against the degree of mixing. In Fig. 1 we plot $N(\rho_\varepsilon)$, $C(\rho_\varepsilon)$, $\mathcal{E}(\rho_\varepsilon,\sigma_m)$, and $\mathcal{F}(\rho_\varepsilon,\sigma_p)$ as given by Eqs. (18, 24, 31, and 48), respectively, against the degree of mixing (linear entropy), Eq. (16), for the family of states determined by ε. An important feature is that for each ε, there is no crossing between the measures, which implies that a well-defined ordering can be given for the respective state. Thus, let ρ_ε^0 be the initial quantum state under a phase damping channel. Then always $\mathcal{E}(\rho_\varepsilon,\sigma_m) \leq N(\rho_\varepsilon) \leq \mathcal{F}(\rho_\varepsilon,\sigma_p) \leq C(\rho_\varepsilon)$ for a given ε. On the other hand let us consider the whole family of states $\rho_\varepsilon(t)$ for a given linear entropy. Then always $N(\rho_\varepsilon) \leq \mathcal{F}(\rho_{\varepsilon'},\sigma_p) \leq C(\rho_{\varepsilon''})$ for any ε, ε', and ε''. But for a given degree of mixing, there is no obvious ordering between $\mathcal{E}(\rho_\varepsilon,\sigma_m)$ and $N(\rho_{\varepsilon'})$ (see the discussion bellow however). Now in Fig. 2 we compare the entanglement measures together with the linear entropy against the fidelity to the pure state, $\mathcal{F}(\rho_{\varepsilon'},\sigma_p)$. Here for a given ε it is observed $\mathcal{E}(\rho_\varepsilon,\sigma_m) \leq N(\rho_\varepsilon) \leq C(\rho_\varepsilon)$. But for a fixed fidelity no obvious ordering between $\mathcal{E}(\rho_\varepsilon,\sigma_m)$ and $N(\rho_{\varepsilon'})$ is observed. In fact the only possible relation is that for a given fidelity (or linear entropy) then $\mathcal{E}(\rho_\varepsilon,\sigma_m) \leq N(\rho_{\varepsilon'})$, for $\varepsilon \leq \varepsilon'$, which is however a very weak relation since there are many $\varepsilon > \varepsilon'$ that also satisfy this inequality. Thus no ordering can be identified for $\mathcal{E}(\rho_\varepsilon,\sigma_m)$ and $N(\rho_{\varepsilon'})$.

VI. CONCLUDING REMARKS

The present work was motivated by the need to analyze the degree of entanglement against mixing for a specific dynamical system composed of two coupled qubits under a phase damping channel. We have discussed the ordering of some possible entanglement measures for a family of pure initial states. We have considered as entanglement measures the negativity N and concurrence (entanglement of formation) C, that is calculated analytically, and the Fidelity $\mathcal{F}$ and Bures distance entanglement measure $\mathcal{E}$, which is calculated numerically and analytically for some case. For the fidelity $\mathcal{F}$ and Bures distance entanglement measure $\mathcal{E}$ it is necessary to define a reference state to compute the measures. The results have shown that the dynamics of the system restricts the possibilities and determine the reference pure state associated to the specific dynamics of the system. In the case of a mixed reference state, we have discussed that the maximally mixed state is not the best choice. Instead it is necessary to choose a mixed state associated to the dynamics of the system. Then the model with phase damping channel has suggested that the best reference states are always associated to the dynamics of the system.

We have shown an entanglement measures ordering for a family of initial states. As the considered entanglement measures are strongly dependent on the initial state and reference state, the measures ordering was then determined for definite values of ε. These results have shown that the Bures distance can be envisaged as a possible quantitative and qualitative measure of entanglement.

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[1] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[2] D. Bouwmeester J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature(London) **390**, 575 (1997).
[3] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[4] K. Mattle, H. Weinfurter, P.G. Kwiat, and A. Zeilinger, Phys. Rev. Lett. **76**, 4656 (1996).
[5] M. Nielsen and I. Chuang, Quantum Computation and quantum Information (Cambridge University Press, Cambridge, England, 2000).
[6] W.J. Munro, D.F.V. James, A.G. White and P.G. Kwiat, Phys. Rev. A **64**, 030302(R) (2001).
[7] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[8] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[9] Tzu-Chieh Wei, et al., Phys. Rev. A **67**, 022110 (2003).
[10] N.A. Peters, T.C. Wei, and P.G. Kwiat, Phys. Rev. A **70**, 052309 (2004).
[11] M. C. de Oliveira and W. J. Munro, Phys. Lett. A **320**, 352 (2004).
[12] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[13] C Monroe, D. M. Meekhof, B. E. King, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **75**, 4714 (1995).
[14] M. Sasura and V. Buzek, J. Mod. Optics **49**, N.10, 1593 (2002).
[15] F. Schmidt-Kaler et al., Nature **422**, 408 (2003).
[16] M. Murao and P. L. Knight, Phys. Rev. A **58**, 663 (1998).
[17] D. P. DiVincenzo, Phys. Rev. A **51**, 1015 (1995).
[18] F. Verstraete, K. Audenaert and B. De Moor, Phys. Rev. A **64**, 012316 (2001)
[19] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
[20] D. Bures, Trans. Am. Math. Soc. **135**, 199 (1969); R. Josza, J. Mod. Opt. **41**, 2315 (1994); M. Hillery, Phys. Rev. A **35**, 725 (1987);
[21] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).
[22] P. Marian, T.A. Marian, and H. Scutaru, Phys. Rev. A **68**, 062309 (2003).
[23] M. C. de Oliveira, Phys. Rev. A **72**, 012317 (2005).

Figure Captions

Fig. 1 - Entanglement measures against linear entropy for a range of ρ_ε^0 initial states. $\varepsilon = 0.5$ (solid line), $\varepsilon = 0.42$ (dashed line), $\varepsilon = 0.37$ (dotted line), $\varepsilon = 0.32$ (dash-dotted line), $\varepsilon = 0.26$ (short-dashed line), $\varepsilon = 0.19$ (short-dotted line), $\varepsilon = 0.12$ (short-dash-dotted line), and $\varepsilon = 0.05$ (dash-dotted-dotted line).

Fig. 2 - Entanglement and mixing measures against pure state fidelity for a range of ρ_ε^0 initial states. $\varepsilon = 0.5$ (solid line), $\varepsilon = 0.42$ (dashed line), $\varepsilon = 0.37$ (dotted line), $\varepsilon = 0.32$ (dash-dotted line), $\varepsilon = 0.26$ (short-dashed line), $\varepsilon = 0.19$ (short-dotted line), $\varepsilon = 0.12$ (short-dash-dotted line), and $\varepsilon = 0.05$ (dash-dotted-dotted line)